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Existence and Bifurcation Theorems for Nonlinear Elliptic Eigenvalue Problems on Unbounded Domains*

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We consider nonlinear elliptic eigenvalue problems on unbounded domains $G \subseteq \mathbb{R}^n$. Using an extended Ljusternik–Schnirelman theory we prove the existence of infinitely many eigenfunctions on every sphere in $L^2(G)$. Moreover, we establish that the infimum λ^* of the spectrum of the linearized problem L is always a bifurcation point. In addition, there is an infinity of branches emanating at λ^* from the trivial line of solutions if λ^* belongs to the essential spectrum of L.

1. INTRODUCTION

Consider a nonlinear eigenvalue problem of the form

$$Lu(x) + f(x, u(x)) = \lambda u(x) \tag{1.1}$$

on an unbounded domain $G \subseteq \mathbb{R}^n$, where L is a second-order self-adjoint elliptic differential operator, and where suitable boundary conditions are imposed. Very little seems to be known about the L^2-theory of such problems even for the case $n = 1$. It is the purpose of this paper to show how Ljusternik–Schnirelman theory can be used to shed some light on the question of existence of solutions to (1.1) with prescribed L^2-norm and on the behaviour of the associated eigenvalues, with particular emphasis on the phenomenon of *bifurcation from the essential spectrum.*

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This phenomenon has recently been studied by various workers. In an operator-theoretic setting, an approximation procedure for the construction of solutions branching from an isolated eigenvalue of infinite multiplicity was considered by Heinz [14] and Sarreither [20]. Boundary value problems for ordinary differential equations on the half-line were treated by Stuart [21–23], Chiappinelli and Stuart [10], Küpper [15, 16], Küpper and Riemer [17] under both the Dirichlet and the Neumann boundary conditions. Almost all these papers deal with the question of bifurcation from the lowest point of the continuous spectrum to the "left," i.e., away from the continuous spectrum. Problems with solution branches lying above the continuous spectrum and bifurcating from its infimum were, for the first time, studied in [15]. There it was noticed that bifurcation to the "right" only occurs if the nonlinearity of the problem presents a certain minimal growth. This minimal growth of the nonlinearity again plays an important role in the case of partial differential equations considered here. Moreover, the abstract results of this paper enable us to understand this growth condition in terms of a compact imbedding property.

In the present paper, we consider nonlinearities of the form

$$f(x, \eta) = f_0(x, \eta) + f_1(x, \eta), \tag{1.2}$$

where f_0 is a continuous real-valued function on $G \times \mathbb{R}$ enjoying a "minimal growth" property for x near infinity, and where f_1 may be considered a small perturbation. More specifically, we shall require f_0 to be odd and continuously differentiable with respect to the second variable, with $\partial f_0/\partial\eta$ nonnegative. The "minimal growth" condition will be given in terms of a number $\sigma > 0$ and a nonnegative function w on G such that

$$\int_G w^{-2/\sigma}(x)\, dx < \infty \tag{1.3}$$

and it reads (with c a positive constant):

$$\left| \frac{\partial f_0}{\partial \eta}(x, \eta) \right| \geqslant c\, |\eta|^\sigma\, w(x) \qquad (\eta \in \mathbb{R}). \tag{1.4}$$

The term f_1 is also assumed to be odd in the second variable, and its growth is restricted in such a way that it remains small compared with f_0 for large $x \in G$. However, f_1 does not have to be continuous or locally bounded; it may, as a function of x, present singularities of the type occurring in L^2-functions.

We then establish, for any $r > 0$, the existence of an infinite sequence (u_j^r, λ_j^r) of solutions to (1.1) such that $\|u_j^r\|^2 = \int_G |u_j^r(x)|^2 dx = r$ and such that the functions u_j^r belong to the Sobolev space $W_0^{1,2}(G)$ and show a specific type of decay at infinity. The fundamental idea for this is to introduce a suitable Banach space X, which is compactly imbedded in the Hilbert space $H = L^2(G)$, and to consider the given nonlinear eigenvalue problem as an equation in the dual space X^* of X (which is a space of distributions). This equation can be derived from a variational problem on X, to which Ljusternik–Schnirelman theory is applicable. The level sets $M_r = \{u \in X/\|u\| = \sqrt{2r}\}$ are necessarily unbounded in the normed space X. However, this difficulty can be handled by the use of a variant of Ljusternik–Schnirelman theory which was recently developed by Bongers [7] and in which unbounded level sets are admissible.

In the unperturbed case (i.e., $f_1 = 0$) more can be said about the solutions (u_j^r, λ_j^r). The eigenvalues λ_j^r tend to ∞ as $j \to \infty$, and we have $\lambda_j^r \geqslant \lambda^*$, where λ^* denotes the lowest point of the spectrum of the self-adjoint operator L. Furthermore, when $f(x, \eta) = f_0(x, \eta)$ is a higher-order term in η, we prove that λ^* is a bifurcation point for problem (1.1) and that an infinity of solution "branches" emanates from $(0, \lambda^*) \in X \times \mathbb{R}$ in case λ^* belongs to the *essential* spectrum. (Note that this situation particularly holds if the spectrum of L is purely continuous as it is the case in many important problems on unbounded domains.) The bifurcation result depends heavily on the variational characterization of the eigenfunctions by a minimax principle. As is typical for the variational approach, we do not, of course, construct connected sets of solutions; the word "branch" merely means a set of nontrivial solutions which intersects every sufficiently small sphere and whose members enjoy a common variational characterization. However, for the special case of ordinary differential equations, it was established in [15] that the positive solutions depend continuously on the eigenvalues parameter and, hence, form a branch in the strict sense of the word.

For the sake of clarity we treat the purely operator-theoretic arguments separately. Thus, Section 2 contains a brief account of the results of [7], as far as they are needed here. In Section 3, we consider a uniformly convex Banach space X, compactly imbedded in a Hilbert space H as a dense subspace and two operators $L_1, F: X \to X^*$, where F is a continuous gradient operator and L_1 is a bounded linear operator from X to X^*, stemming from an (unbounded) self-adjoint operator L in H. We then pose the nonlinear eigenvalue problem

$$L_1 u + F(u) = \lambda u \tag{1.5}$$

in X^* (which is possible since X may be considered as a subspace of X^*), and, under suitable assumptions on F, we establish the existence of eigenvectors $u_j^r \in X$ ($j \in \mathbb{N}$) with prescribed Hilbert norm $\sqrt{2r}$ as well as some of their additional properties. The unperturbed case, in which F is monotone, is dealt with first (Theorem 3.1), and the general case is derived from it by an easy compact perturbation argument (Theorem 3.2). An abstract version of the above-mentioned bifurcation result is then proved in Section 4.

This Abstract theory is then applied to second-order partial differential equations (Section 5) and finally to ordinary differential equations (Section 6). Since our emphasis is on the nonlinearity and since we wish not to increase the length of this paper unduly, our assumptions concerning the coefficients of the linear elliptic differential operator L are not chosen to be the most general ones possible.

Roughly speaking, the coefficients are assumed to be bounded and smooth so that the corresponding quadratic form is defined on the Sobolev space $W_0^{1,2}(G)$. However, the results of Sections 3 and 4 can also be applied to elliptic operators of higher order.

For ordinary differential equations we obtain additional results which generalize results by Küpper [15]. In particular, the eigenfunctions u_j^r decrease uniformly as $x \to \infty$, and for $j = 1$ one obtains a unique positive solution. Moreover, condition (1.3) turns out to be necessary for the existence of L^2-solutions of a large class of problems.

2. Preliminaries

Our argument is based on variational methods. Since our applications require the minimisation of functionals on unbounded sets, classical variational methods do not suffice. Here we show that these difficulties can be overcome by using a generalization of the Ljusternik–Schnirelman theory which was recently developed by Bongers [7]. We begin with a brief summary of the main results as far as needed here. For the proof and details we refer to [7].

Let X denote a uniformly convex Banach space of infinite dimension. We shall consider functionals ψ and ρ which satisfy the following hypotheses:

(A) $\psi, \rho: X \to \mathbb{R}$ are even C^1-functionals with $\psi(0) = \rho(0) = 0$. Their gradients $\psi', \rho': X \to X^*$ are uniformly continuous on bounded sets. Further ρ' satisfies $u_n \rightharpoonup_X u \Rightarrow \rho'(u_n) \to_{X^*} \rho'(u)$ ("strong continuity" in the sense of Vainberg [25]).

(B) Assume $r > 0$. For any $v \in X - \{0\}$ there exists a unique $s(v) \in (0, \infty)$ such that $s(v)\, v \in M_r = \{u \in X / \rho(u) = \sqrt{2r}\}$. If $\{u_n\}$ is a bounded sequence in X with $\rho(u_n) \to r$, then $s(u_n) \to 1$. There exists numbers $d(r) > 0$ and $c(r)$ such that $\rho'(u)\, u \geqslant d(r)$, $\psi(u) \geqslant c(r)$ for all $u \in M_r$.

Let Σ denote the set of all closed and symmetric subsets of $X - \{0\}$. For $A \in \Sigma$, we denote by gen(A) the genus of A (in the sense of Coffman [12] for example).

THEOREM 2.1 (Bongers [7]). *Suppose* (A) *and* (B) *hold and assume in addition:*

(1) $u_n \rightharpoonup_X u, \ \psi'(u_n) \to_{X^*} v \Rightarrow u_n \to_X u$,

(2) $\psi^{-1}(J) \cap M_r$ *is bounded for any bounded interval* $J \subseteq \mathbb{R}$.

Then for any $r > 0$

(i) *the numbers* b_j *which are defined by*

$$b_j = \inf_{\substack{A \in \Sigma, A \subseteq M_r \\ \operatorname{gen}(A) \geqslant j}} \sup_{u \in A} \psi(u) \qquad (j = 1, 2, \ldots)$$

are critical values of ψ *on* M_r *and satisfy* $\lim_{j \to \infty} b_j = \infty$.

(ii) *There exists an infinite sequence* $(u_j, \lambda_j) \subseteq M_r \times \mathbb{R}$ *of distinct solutions to the eigenvalue problem* $\psi'(u) = \lambda \rho'(u)$ *such that* $\psi(u_j) = b_j$.

(iii) *If* $b_j = b_{j+1} = \cdots = b_{j+q-1}$, *then* $\operatorname{gen}(K_{b_j}) \geqslant q$, *where* $K_b = \{u \in M_r / \psi(u) = b$ *and* $\psi'(u) = \lambda \rho'(u)$ *for some* $\lambda \in \mathbb{R}\}$.

This is essentially Theorem 1.17 combined with Proposition 3.3 of [7]. The fact that $\lim b_j = \infty$ is not mentioned explicitly in Theorem 1.17 of [7], but its proof follows standard patterns (see, e.g., Rabinowitz [19]).

Remark. When the assumption that ψ and ρ are even is dropped, the existence of at least one critical value b_1 of ψ on each level set M_r can still be proved using the methods of [7]. However, if ψ is convex (which will almost always be the case in our applications), it can be shown very easily that ψ attains its minimum on M_r. One has only to note that under the assumptions of Theorem 2.1 the standard argument based on the weak lower semicontinuity of the convex functional can be carried through and that it leads to a solution $u \in M_r$ of the minimisation problem because ρ is weakly continuous and, hence, M_r is weakly (sequentially) closed in X.

3. EXISTENCE

The results of the preceding section are now applied to nonlinear eigenvalue problems $A(u) = \lambda u$, where A is densely defined in a Hilbert space H. We assume that the nonlinear operator A is the sum of a positive self-adjoint operator L possibly with purely continuous spectrum) and a monotone nonlinearity. The fundamental idea consists in an appropriate choice of the space X for which the Ljusternik–Schnirelman theory of Section 2 can be utilized; in particular we need that X is dense and compactly imbedded in H.

Our results are developed to include nonlinear elliptic eigenvalue problems on unbounded domains as a special case. For a better understanding of the operator-theoretic setting we start with an easy model example which contains the basic difficilties: ·

$$-\Delta u + w(x)\,|u|^{\sigma}\,u = \lambda u \qquad (u \in W_0^{1,2}(G)),$$

where $G \subseteq \mathbb{R}^n$ is a smooth domain, w a positive and continuous function and $\sigma > 0$. As basic Hilbert space we choose $H = L^2(G)$.

In the case of a bounded domain G the choice $X = W_0^{1,2}(G)$ provides the compact imbedding in $H = L^2(G)$ by Sobolev's theorem. In the case of an unbounded domain the imbedding theorem fails and we then need the nonlinear part of our equation to ensure the compact imbedding. For example, the functional corresponding to the above equation is given by

$$\psi(u) = \frac{1}{2}\int_G |\mathrm{grad}\, u|^2\,(x)\,dx + \frac{1}{\sigma+2}\int_G w(x)\,|u|^{\sigma+2}\,(x)\,dx.$$

While the quadratic part of ψ is defined on $W_0^{1,2}(G)$ the second part is defined on the weighted space $Y = L^p(w\,dx)$ with $p = \sigma + 2$. An easy calculation shows that the intersection $X = W_0^{1,2} \cap Y$ is compactly imbedded in H provided that w satisfies the growth condition $\int_G w^{-2/\sigma}(x)\,dx < \infty$.

Let H be a Hilbert space with inner product $\langle\, ,\rangle$ and norm $\|\ \|$ and assume that T is a densely defined closed linear operator. Since T is closed, its domain of definition $D(T)$ is a Hilbert space with respect to the graph norm $\|u\|_T = \{\|u\|^2 + \|Tu\|^2\}^{1/2}$ of T. We denote this Hilbert space by H_T. Further, let $(Y, \|\ \|_Y)$ be a uniformly convex Banach space which is a vector subspace of H such that $H_T \cap Y \neq \{0\}$.

LEMMA 3.0. *$X = H_T \cap Y$ is a uniformly convex Banach space with the norm $\|u\|_X = \{\|u\|_T^2 + \|u\|_Y^2\}^{1/2}$.*

Proof. Let Z be the product space $H_T \times Y$ endowed with the norm $\|(u, v)\|_Z = (\|u\|_T^2 + \|v\|_Y^2)^{1/2}$. Evidently the map $J: X \to Z$ defined by $J(u) = (u, u)$ is an isometric isomorphism of X onto a close subspace of the Banach space Z. Finally, we remark that Z is uniformly convex by a classical theorem of Day [13], which ensures the uniform convexity of X.

To formulate our equation precisely let T_1 be the bounded operator that arises when we consider the restriction of T to $X \subseteq H_T$ as an operator from X to H. Its dual operator T_1^* then operates from $H^* = H$ into X^*, and obviously $T_1^* T_1$ coincides with the self-adjoint operator T^*T on $X \cap D(T^*T)$ if we consider H as a subspace of X^*. The spectrum of T^*T shall be denoted $\sigma(T^*T)$. Finally, we take F as a nonlinear mapping from X to X^*.

As our basic problem we consider the existence of solutions of the equation

$$T_1^* T_1 u + F(u) = \lambda u \qquad (u \in X). \tag{3.1}$$

In application to differential equations this will lead to weak solutions, but these solutions have considerable regularity properties since they lie in X and not just X^*.

We shall need the following hypotheses:

(I) The Banach space $(X, \| \|_X)$ is infinite dimensional, campactly imbedded and dense in H.

(II) The nonlinearity $F: X \to X^*$ is a monotone odd operator and the gradient of a C^1-functional $\phi: X \to \mathbb{R}$ with $\phi(0) = 0$. Further, F satisfies:

(1) F is uniformly continuous on bounded sets.

(2) $u_n \rightharpoonup_X u$ and $\langle F(u_n) - F(u), u_n - n \rangle \to 0 \Rightarrow u_n \to_Y u$ (where $\langle \, , \rangle$ denotes the pairing between X and its dual space X^*).

(3) There exists a measurable function $\gamma: [0, \infty) \to [0, \infty)$ with $\int_0^\infty t^{-1} \gamma(t) \, dt = \infty$ and $\langle F(u), u \rangle \geqslant \gamma(\|u\|_Y)$ $(u \in X)$.

Hypotheses (I) and (II) imply that Eq. (3.1) stems from a variational problem, i.e., it can be written as

$$\psi'(u) = \lambda \rho'(u) \tag{3.2}$$

with functionals $\psi, \rho: X \to \mathbb{R}$ defined by

$$\psi(u) = \|Tu\|^2/2 + \phi(u), \tag{3.3}$$

$$\rho(u) = \|u\|^2/2. \tag{3.4}$$

THEOREM 3.1. *Let* (I) *and* (II) *be satisfied. Then the functionals* ψ *and* ρ *given by* (3.3) *and* (3.4) *satisfy all the hypotheses of Theorem* 2.1. *Consequently we have:*

(i) *on each sphere* $M_r = \{u \in X / \|u\| = \sqrt{2r}\}$ *there exist infinitely many distinct solutions* (u_j, λ_j) $(j = 1, 2, \ldots)$ *of* (3.1),

(ii) $$b_j = \psi(u_j) = \inf_{\substack{A \in \Sigma, A \subseteq M_r \\ \text{gen}(A) \geqslant j}} \sup_{u \in A} \psi(u), \tag{3.5}$$

(iii) $\lim_{j \to \infty} b_j = \infty$,

(iv) $\lim_{j \to \infty} \lambda_j = \infty$,

(v) $\lambda_j \geqslant \lambda^* = \min \sigma(T^*T)$ $(j = 1, 2, \ldots)$.

Remarks. (i) Condition (II(2)) is a variant of Browder's (S)-condition (cf. [9]). It is satisfied for example, when F is of the form $F = F_1 + F_2$, where F_1 satisfies

$$\langle F_1(u) - F_1(v), u - v \rangle \geqslant \beta(\|u - v\|_Y) \qquad (u, v \in X)$$

with a continuous function β: $[0, \infty) \to [0, \infty)$ such that $\beta(0) = 0$, $\beta(t) > 0$, for $t > 0$ and $\liminf_{t \to \infty} \beta(t) > 0$, and where F_2 is strongly continuous from X to X^* (i.e., $u_n \rightharpoonup_X u \Rightarrow F_2(u_n) \to_{X^*} F_2(u)$).

(ii) If the assumption that F should be odd is dropped, we still obtain solutions (u, λ) such that $u \in M_r$, $\lambda \geqslant \lambda^*$ and

$$\psi(u) = \min_{v \in M_r} \psi(v)$$

for r arbitrary. This is evident from the proof of Theorem 3.1 and the remark following Theorem 2.1.

(iii) The assumption of monotonicity will be relaxed later in this section by a perturbation argument.

Proof of Theorem 3.1. We first show that the hypotheses of Theorem 2.1 hold. The functionals ρ and ψ are Fréchet-differentiable and satisfy $\rho(0) = \psi(0) = 0$. By construction their gradients are given by $\rho'(u) = u$, $\psi'(u) = A(u) = T_1^* T_1 u + F(u)$. Since X is imbedded in H_T and H we have for all u, v, $h \in X$ and some constant $M\, |\langle T_1^* T_1(u - v), h\rangle| \leqslant |\langle T_1(u - v), T_1 h\rangle| \leqslant \| u - v \|_T \; \| h \|_T \leqslant M \| u - v \|_X \; \| h \|_X$. Together with hypothesis (II(1)) this implies that ρ' and ψ' are uniformly continuous on bounded sets. The gradient $\rho'(u) = u$ satisfies (A) since X is compactly imbedded in H. Hence, hypothesis (A) is satisfied, while (B) obviously holds with $s(v) = \sqrt{2r}/\| v \|$, $d(r) = 2r$, $c(r) = 0$.

LEMMA 3.1. *$A = \psi'$ satisfies* (1) *of Theorem* 2.1.

Proof. Assume $u_n \rightharpoonup u$ and $A(u_n) \to v$. We first show that $\langle A(u_n) - A(u), u_n - u\rangle \to 0$ since $0 \leqslant \langle A(u_n) - A(u), u_n - u\rangle = \langle A(u_n) - v, u_n - u\rangle + \langle v - A(u), u_n - u\rangle \to 0$ and using $A = T_1^* T_1 + F$ we have $0 \leqslant \| T_1(u_n - u)\|^2 + \langle F(u_n) - F(u), u_n - u\rangle = \langle A(u_n) - A(u), u_n - u\rangle \to 0$. Since F is monotone we have $\| T_1(u_n - u)\| \to 0$ and $u_n \to u$ in Y by $(II(2))$ and consequently $u_n \to u$ in X.

LEMMA 3.2. *If $r > 0$ and $J \subseteq \mathbb{R}$ is a bounded interval, then $N_{r,J} = \{u \in X/\| u \| \leqslant r, \psi(u) \in J\}$ is bounded in X. In particular, we have* (2) *of Theorem* 2.1.

Proof. Fix $r > 0$ and a bounded interval $J \subseteq \mathbb{R}$. There exists a constant $b \geqslant 0$ such that $0 \leqslant \psi(u) \leqslant b$ for all $u \in N_{r,J}$, hence in particular $\| T_1 u \|^2 \leqslant 2b$ and $b \geqslant \phi(u) = \int_0^1 \langle F(tu), u\rangle\, dt = \int_0^1 t^{-1}\langle F(t), tu\rangle\, dt \geqslant \int_0^1 t^{-1}\gamma(t \| u \|_Y)\, dt = \int_0^{\| u \|_Y} t^{-1}\gamma(t)\, dt$. Thus, there exists a constant M such that $\| u \|_Y \leqslant M$; consequently $\| u \|_X^2 \leqslant M^2 + 2b + 2r$.

Thus the hypotheses of Theorem 2.1 are satisfied and (i), (ii) and (iii) are established. To prove (iv), we use the monotonicity of F which implies that

$\langle F(tu), u\rangle \leqslant \langle F(u), u\rangle$ for all $u \in X$ and $t \in [0, 1]$. Hence, $b_j = \psi(u_j) = \|T_1 u_j\|^2/2 + \int_0^1 \langle F(tu_j), u_j\rangle\, dt \leqslant \|T_1 u_j\|^2/2 + \langle F(u_j), u_j\rangle \leqslant \langle A(u_j), u_j\rangle = \lambda_j \|u_j\|^2$ and, thus, $\lambda_j \geqslant b_j/(2r)$. To prove (v) note that $\langle F(u), u\rangle \geqslant 0$ for every $u \in X$, and that for $u \in D_{T^*T}$ we have $\lambda^* \|u\|^2 \leqslant \langle T^*Tu, u\rangle = \|Tu\|^2$. Since D_{T^*T} is dense in the Hilbert space H_T, it follows that $\lambda^* \|u\|^2 \leqslant \|Tu\|^2$ for all $u \in H_T$. Thus, for $u \in X \subseteq H_T$ we have $\langle A(u), u\rangle = \langle T_1^* T_1 u, u\rangle + \langle F(u), u\rangle \geqslant \langle T_1 u, T_1 u\rangle = \|Tu\|^2 \geqslant \lambda^* \|u\|^2$ from which our assertion follows. This completes the proof of Theorem 3.1.

To conclude this section, we give an extension of Theorem 3.1, allowing "compact perturbations" of the nonlinearities considered so far.

THEOREM 3.2. *Let $A = T_1^* T_1 + F$ satisfy the assumptions of Theorem* 3.1, *and let $F_1: X \to X^*$ be an odd operator which has the following properties:*

(i) *F_1 is strongly continuous (in the sense of Vainberg* [25]),

(ii) *F_1 is the Gateaux derivative of a functional $\phi_1: X \to \mathbb{R}$,*

(iii) *ϕ_1 is bounded below on any set $B \subseteq X$ which is bounded in H.*

Then, assertions (i)–(iii) *of Theorem* 3.1 *are also valid for the problem*

$$A(u) + F_1(u) = \lambda u, \qquad u \in X. \tag{3.6}$$

Proof. We shall apply Theorem 2.1 to the functionals ρ and ψ_1, where $\rho(u) = \|u\|^2/2$, as before, and where

$$\psi_1 = \psi + \phi_1,$$

ψ being the potential of A as before, and ϕ_1 being chosen such as to satisfy $\phi_1(0) = 0$. From the proof of Theorem 3.1 we already know that X, ψ and ρ satisfy the assumptions of Theorem 2.1.

Moreover, classical results on operators in reflexive Banach spaces show that under our assumptions F_1 must be uniformly continuous on bounded subsets of X and must be the Fréchet derivative of ϕ_1 (cf. Vainberg [25]). Since from assumption (iii) and the positivity of ψ it also follows that ψ_1 is bounded below on M_r for any $r > 0$, we see that (A) and (B) are satisfied by X, ρ and ψ_1. It remains to prove that ψ_1 satisfies conditions (1) and (2) from Theorem 2.1. But (1) follows immediately from the fact that ψ satisfies (1) together with the strong continuity of F_1, and for (2), we only have to note that for any $r > 0$ and any compact interval $J = [a, b]$ one has

$$M_r \cap \psi_1^{-1}(J) \subseteq M_r \cap \psi^{-1}([0, b - c(r)]),$$

where $c(r)$ is some lower bound for ϕ_1 on M_r. Since ψ satisfies (2), this yields the boundedness of $M_r \cap \psi_1^{-1}(J)$, which ends the proof.

Remark. Under the additional assumption that for some constant $c_0 > 0$ we have $\langle F_1(tu), u\rangle \leqslant c_0\langle F_1(u), u\rangle$ for $u \in X$ and $0 \leqslant t \leqslant 1$, one can also easily establish assertion (iv) of Theorem 3.1 for the problem (3.6). Note that the above condition is satisfied, for example, when F_1 is monotone or homogeneous.

4. Bifurcation

Conditions (I) and (II) guarantee the existence of infinitely many solutions of (3.1) on every sphere. In this section we show how a bifurcation result can be proved under slight additional assumptions by using the inf-sup-characterization of the critical values. To be precise, we shall require the following bifurcation hypotheses.

(III) Bifurcation hypotheses.

(1) $\|F(u)\|_{X^*} = o(\|u\|_X)$ for $\|u\|_X \to 0$.

(2) There exist constants $\delta > 0$, $C_1 > 0$ such that $\langle F(u), u\rangle \leqslant C_1\phi(u)$ for any $u \in X$ with $\|u\| < \delta$, $\psi(u) < \delta$.

(3) $X \cap D_L$ is dense in D_L.

Here and in the sequel D_L denotes the domain of the positive self-adjoint operator $L = T^*T$, and D_L is always equipped with the graph norm generated by L. (The density required in (III(3)) is understood with respect to this norm.)

Condition (III(1)) says that F is a nonlinearity of higher order. Condition (III(2)) is satisfied, for instance, by sums of finitely many homogeneous terms, but it will become apparent in the applications (especially in Section 6) that there are much wider classes of operators F satisfying (III(2)). Moreover, in applications to differential equations on a domain $G \subseteq \mathbb{R}^n$ the space X will usually contain a set of smooth functions which is dense in D_L, and, hence, condition (III(3)) will be satisfied under slight regularity assumptions.

THEOREM 4.1. *Suppose* (I), (II), (III) *are satisfied. Then the solution sequences* $(u_j^r, \lambda_j^r) \subseteq M_\rho \times [\lambda^*, \infty)(\rho = r^2/2)$ *constructed in Theorem* 3.1 *have the following properties.*

(i) *If λ^* is an isolated eigenvalue of L of finite multiplicity m, then for $j = 1,\ldots, m$:*

$$\lim_{r\to 0+} u_j^r = 0 \textit{ strongly in } X, \tag{4.1}$$

$$\lim_{r\to 0+} \lambda_j^r = \lambda^*. \tag{4.2}$$

(ii) *If λ^* belongs to the essential spectrum of L, then* (4.1) *and* (4.2) *hold for all* $j = 1, 2,\ldots$

In particular, λ^* is alwas a bifurcation point for problem (3.1).

Proof. Let $r > 0$. By Theorem 3.1 we know that the solutions $u_j^r \in M_\rho$ are critical points of $\psi | M_\rho$ corresponding to the critical values

$$b_j^r = \inf_{\substack{A \in \Sigma \\ A \subseteq M_\rho \\ \operatorname{gen}(A) \geqslant j}} \sup_{u \in A} \psi(u) \qquad (j \geqslant 1).$$

To prove the limit relations (4.1), (4.2), we shall assume $\lambda^* = 0$, which is evidently no loss of generality. We first need a lemma.

LEMMA 4.1. (a) *If $\lambda^* = 0$ is an isolated eigenvalue of L of finite multiplicity m, then for any $\varepsilon > 0$ there exists an m-dimensional linear subspace Z_m of X such that for any $u \in Z_m$ we have $\|Tu\|^2 \leqslant \varepsilon \|u\|^2$.*

(b) *If $\lambda^* = 0$ belongs to the essential spectrum of L, the above is true even for every $m \in \mathbb{N}$.*

Proof. Let $H_{\mathbb{C}} = H \oplus iH$ and $L_{\mathbb{C}}$ be the complexifications of H and L, respectively. Then $L_{\mathbb{C}}$ is a self-adjoint operator in $H_{\mathbb{C}}$, and we may consider its special resolution $(E_\lambda)_\lambda$. It is readily verified that the E_λ are real operators (for example, the uniqueness of the spectral resolution shows that E_λ commutes with complex conjugation in $H_{\mathbb{C}}$), and this implies that $E_\lambda(H) \subseteq H$ and that $E_\lambda(H_{\mathbb{C}}) = E_\lambda(H) \oplus iE_\lambda(H)$.

For a given $\varepsilon > 0$, choose now δ such that

$$0 < \delta < \min(1/2, \varepsilon/8)$$

and consider the subspace $E_\delta(H)$ of $D_L = D_{T^*T}$. In case (a), the (real) dimension of $E_\delta(H)$ is $\geqslant m$ because the eigenvectors corresponding to $\lambda^* = 0$ belong to $E_\delta(H)$, and, hence, we may choose m orthonormal vectors $a_1,\ldots, a_m \in E_\delta(H)$. In case (b) the space $E_\delta(H)$ is infinite-dimensional, and so we may choose orthonormal vectors $a_1,\ldots, a_m \in E_\delta(H)$ for any $m \in \mathbb{N}$. By (III(3)), we may also choose $e_1,\ldots, e_m \in X \cap D_L$ such that the e_j $(j = 1,\ldots, m)$ are still linearly independent and $\|a_j - e_j\|_L < \delta/\sqrt{m}$ $(j = 1,\ldots, m)$, where $\|\cdot\|_L$ denotes the graph norm induced by L. Let Z_m be the linear span of $\{e_1,\ldots, e_m\}$. Then Z_m is evidently an m-dimensional linear subspace of $X \cap D_L$. Let $v \in Z_m$. We have $v = \sum_{k=1}^m \zeta_k e_k$, where $\zeta_1,\ldots,\zeta_m \in \mathbb{R}$, and put $u = \sum_{k=1}^m \zeta_k a_k$. Since the a_k are orthonormal, we have $\|u\|^2 = \sum_{k=1}^m |\zeta_k|^2$, and hence $\|u - v\|_L \leqslant \sum_{k=1}^m |\zeta_k| \, \|a_k = e_k\|_L \leqslant \sqrt{m}(\sum_{k=1}^m |\zeta_k|^2)^{1/2} \max_{1 \leqslant k \leqslant m} \|a_k - e_k\|_L < \sqrt{m} \cdot \|u\| \cdot \delta/\sqrt{m} = \delta \|u\|$. This implies $\|u\| \leqslant$

$\|v\| + \|u - v\| \leqslant \|v\| + \|u - v\|_L \leqslant \|v\| + \delta \|u\|$, and since $0 < \delta < 1/2$, we obtain

$$\|u\| \leqslant \frac{1}{1-\delta} \|v\| \leqslant 2 \|v\|.$$

Since $u \in E_\delta(H)$ and $0 = \lambda^* = \min \sigma(L)$, spectral theory tells us that $0 \leqslant \langle Lu, u \rangle \leqslant \delta \|u\|^2$ and $\|Lu\| \leqslant \delta \|u\|$, and we finally obtain:

$$\begin{aligned} \|Tv\|^2 &= \langle Lv, v \rangle, \\ &\leqslant |\langle Lv - Lu, v \rangle| + |\langle Lu, v - u \rangle| + |\langle Lu, u \rangle|, \\ &\leqslant \|v - u\|_L \|v\| + \|Lu\| \|v - u\|_L + \delta \|u\|^2, \\ &\leqslant \delta \|u\| \|v\| + \delta^2 \|u\|^2 + \delta \|u\|^2 \leqslant (6\delta + 4\delta^2) \|v\|^2, \\ &\leqslant 8\delta \|v\|^2 \leqslant \varepsilon \|v\|^2. \end{aligned}$$

Next, we apply this lemma to show

$$\lim_{r \to 0+} \psi(u_j^r) r^{-2} = \lim_{r \to 0+} b_j^r r^{-2} = 0 \tag{4.3}$$

for $1 \leqslant j \leqslant m$ in case (a) and j arbitrary in case (b) of the lemma. Let $\varepsilon > 0$, and choose a space Z_m as in Lemma 4.1. From (III(1)) we infer that $\phi(u) = o(\|u\|_X^2)$ for $\|u\|_X \to 0$. But on the finite-dimensional space Z_m all norms are equivalent, hence we also have $\phi(u) = o(\|u\|^2)$ for $\|u\| \to 0$ and $u \in Z_m$. Therefore, we can choose $r_0 > 0$ such that $\phi(u) < \varepsilon \|u\|^2/2$ for $u \in Z_m$, $\|u\| < r_0$. Let now $0 < r < r_0$ and again $\rho = r^2/2$. The genus of the set $A_0 = Z_m \cap M_\rho$ is m, and hence the definition of b_j^r implies $0 \leqslant b_j^r \leqslant \sup_{u \in A_0} \psi(u) \leqslant \sup_{u \in A_0} \{\|Tu\|^2/2\} + \sup_{u \in A_0} \phi(u) \leqslant \varepsilon\rho + \varepsilon\rho = \varepsilon r^2$. This yields $0 \leqslant b_j^r r^{-2} \leqslant \varepsilon$ for $0 < r < r_0$, i.e., we have proved (4.3).

To prove (4.1), consider a null sequence of numbers $r_n > 0$. Fix j as above, and let us write $u_n = u_j^{r_n}$ for the moment. From (4.3) it follows in particular that

$$\lim_{n \to \infty} \psi(u_n) = 0. \tag{4.4}$$

By Lemma 3.2 from the proof of Theorem 3.1 we may conclude that the sequence $\{u_n\}$ is bounded in X. The space X is reflexive (due to its uniform convexity) and, thus, the sequence $\{u_n\}$ must have weakly convergent subsequences. If $u_{n_k} \rightharpoonup u^*$ in X, then it follows that $u_{n_k} \to u^*$ in H because the imbedding $X \to H$ is compact. But $\|u_{n_k}\| = r_{n_k} \to 0$, hence $u^* = 0$. Thus, 0 is the common limit of all weakly convergent subsequences of $\{u_n\}$, from which it follows that $u_n \rightharpoonup 0$ in X.

Using (4.4), the positivity assumptions and the definition of ψ, we see that

$$\|Tu_n\|^2 \to 0, \tag{4.5}$$

$$\phi(u_n) \to 0 \tag{4.6}$$

for $n \to \infty$. From (4.5) it follows that $\{u_n\}$ converges strongly to 0 in H_T, and from (4.6) we obtain $\langle F(u_n), u_n \rangle \to 0$ $(n \to \infty)$ by virtue of (III(2)) and (4.4). Applying hypothesis (II(2)), we infer that $\lim \|u_n\|_Y = 0$. The definition of the norm now shows at once that we have (4.1).

Finally, put $\lambda_n = \lambda_j^{r_n}$. We have to show that $\lim_{n\to\infty} \lambda_n = \lambda^* = 0$. By (4.4), we may again apply (III(2)) and, thus, suppose that $0 \leqslant \langle F(u_n), u_n \rangle \leqslant C_1 \phi(u_n)$ for every n. From $A(u_n) = \lambda_n u_n$ we obtain $0 \leqslant \lambda_n r_n^2 = \langle A(u_n), u_n \rangle = \|Tu_n\|^2 + \langle F(u_n), u_n \rangle \leqslant (2 + C_1)\{\|Tu_n\|/2 + \phi(u_n)\} = (2 + C_1)\, \psi(u_n)$ and, hence, $0 \leqslant \lambda_n \leqslant r_n^{-2} \psi(u_n)$. Thus, (4.3) yields the desired result, and the proof of Theorem 4.1 is completed.

5. Application to Elliptic Eigenvalue Problems

We now apply the general theorems 3.1, 3.2 and 4.1 to nonlinear elliptic eigenvalue problems in unbounded domains. Since the abstract theory has shown that the existence of nontrivial solutions essentially depends on the behaviour of the nonlinearity, we do not try for the utmost generality concerning the linear part but rather concentrate on the nonlinearity.

Let $G \subseteq \mathbb{R}^n$ be a bounded or unbounded domain and consider the eigenvalue problem

$$\mathscr{L}u + f(x, u) = \lambda u, \tag{5.1}$$

$$u \mid \partial G = 0, \qquad u \in L^2(G). \tag{5.2}$$

Here $\mathscr{L}$ is a linear uniformly elliptic differential operator in C given by

$$\mathscr{L}u = -\sum_{j,k=1}^{n} (P_{jk}(x)\, u_{x_k})_{x_j} + Q(x)\, u \tag{5.3}$$

with bounded real coefficients $P_{jk} \in C^1(\bar{G})$, $Q \in L^\infty(G)$ such that $P_{jk} = P_{kj}$ for any $j, k = 1, \ldots, n$. The nonlinearity f is assumed to be of the form

$$f(x, \eta) = \int_0^\eta g(x, \xi)\, d\xi, \tag{5.4}$$

where $g: G \times \mathbb{R} \to [0, \infty)$ is a continuous function which is even in the second variable and satisfies

(F1) $\delta w(x) |\eta|^\sigma \leqslant g(x, \eta) \leqslant w(x) |\eta|^\sigma + g_1(x, \eta)$

for some constants $\delta > 0$, $\sigma > 0$ and continuous functions $w: G \to [0, \infty)$, $g_1: G \times \mathbb{R} \to [0, \infty)$.

To fit the problem (5.1), (5.2) into the abstract setting we choose for H the real Hilbert space $L^2(G)$ with the usual inner product and norm. Since $\mathscr{L}$ is uniformly elliptic and has bounded coefficients, the quadratic form $\int_G (\sum_{j,k=1}^n P_{jk}(x)\, u_{x_j}(x)\, u_{x_k}(x) + Q(x)\, u^2(x))\, dx$ is defined and closed on the Sobolev space $W_0^{1,2}(G)$ and hence gives rise to a unique semibounded, self-adjoint extension L of $\mathscr{L}$ with domain D_L contained in $W_0^{1,2}(G)$. Without loss of generality we assume L to be positive. Then we can write $L = T^*T$ with $T = L^{1/2}$ and $H_T = W_0^{1,2}(G)$. (Note that the graph norm of T is equivalent to the usual norm on $W_0^{1,2}(G)$.)

The space Y depends on the constant $\sigma > 0$ and the function w appearing in (F1). It is chosen to be the weighted Lebesgue space $L^p(G, w\, dx)$ with $p = \sigma + 2$. (A notation which shall be used throughout the rest of the paper.) It is well known that this space is a uniformly convex Banach space. In keeping with Section 3 we define $X = W_0^{1,2}(G) \cap Y$ with the norm

$$\|u\|_X = \left\{ \int_G (|\operatorname{grad} u|^2 (x) + u^2(x))\, dx + \left(\int_G w(x)\, |u|^p (x)\, dx \right)^{2/p} \right\}^{1/2}.$$

In order to apply the general results we need the following hypotheses:

(F2) $\int_G w^{-2/\sigma}(x)\, dx < \infty$.

(F3) There exists a continuous function Ω: $[0, \infty) \to [0, \infty)$ such that $\Omega(0) = 0$ and such that for any $u, v, h \in X$ we have

$$\int_G g_1(x, h(x))\, |u(x)\, v(x)|\, dx \leqslant \Omega(\|h\|_X)\, \|u\|_X\, \|v\|_X. \tag{5.5}$$

The growth condition (F2) is essential for the space X to be compactly imbedded in H, as we shall soon see. The somewhat technical condition (F3) is designed to include a large variety of cases. Explicit sufficient conditions for (F3) can easily be derived using the generalized Hölder inequality together with the continuous imbedding $X \to L^p(G, w\, dx)$ and the Sobolev imbedding $W_0^{1,2}(G) \to L_q(G)$ for suitable q depending on the dimension n. To illustrate this, consider

EXAMPLE 5.1. Suppose g_1 is of the form

$$g_1(x, \eta) = \sum_{j=1}^{m} w_j(x) |\eta|^{\tau_j}$$

with continuous nonnegative functions w_j and constants $\tau_j > 0$ $(j = 1,..., m)$. Suppose that the following integrals are finite:

$$C_j = \int_G (w_j^p w^{-\tau_j - 2})^{1/(\sigma - \tau_j)} (x)\, dx \qquad \text{in case } t_j < \sigma$$

and

$$C_j = \int_G (w_j^p(x)\, w^{-2}(x))^{1/(\sigma - \tau_j/\gamma)}\, dx \qquad \text{in case } \sigma < \tau_j < \sigma \cdot \gamma,$$

where $\gamma = 2n/(p(n-2))$ for $n \geqslant 3$ and γ arbitrary for $n \in \{1, 2\}$. Then (F3) holds with $\Omega(t) = C \sum_{j=1}^{m} C_j t^{\tau_j}$ where $C > 0$ is some constant. To prove this, we write

$$\int_G w_j |h|^{\tau_j} |u|\, |v|\, dx$$

$$= \int_G (w_j w^{-(\tau_j + 2)/p})(w^{\tau_j/p} |h|^{\tau_j})(w^{1/p} |u|)(w^{1/p} |v|)\, dx$$

in the case $\tau_j < \sigma$ and

$$\int_G w_j |h|^{\tau_j} |u|\, |v|\, dx = \int_G (w_j w^{-2/p})(|h|^{\tau_j})(w^{1/p} |u|)(w^{1/p} |v|)\, dx$$

in the case $\sigma < \tau_j < \sigma\gamma$ and use the Hölder inequality for four factors together with the continuous imbedding $X \to Y$. For the second case we also invoke the Sobolev inequality for the second factor.

THEOREM 5.1. *Suppose that* (F1), (F2), (F3) *are satisfied. Then, for each* $r > 0$ *there exists an infinite sequence* (u_j, λ_j) $(j = 1, 2,...)$ *of* (*weak*) *solutions of* (5.1), (5.2) *such that* $\|u_j\|^2 = 2r$. *For these solutions all the assertions of Theorem* 3.1 *hold.*

Remark. These functions u_j are in $C^2(G)$ and hence classical solutions of (5.1) if the following additional conditions are satisfied. The coefficients Q and P_{jk} along with the first derivatives of the P_{jk} $(j, k = 1,..., n)$ are locally Hölder continuous in G, the function f is locally Hölder continuous in

$G \times \mathbb{R}$, and for each compact subset $K \subseteq G$ there is a constant $C_K > 0$ such that for all $\eta \in \mathbb{R}$ $\sup_{x \in K} |g(x, \eta)| \leqslant C_K |\eta|^s$, where $0 < s < 4/(n-2)$ in case $n \geqslant 3$ respectively s arbitrary in case $n = 2$. This can be proved by using standard regularity arguments in a compact neighbourhood of an arbitrary point $x_0 \in G$.

COROLLARY 5.1. *The assertions of Theorem* 5.1 (*with the exception of the results* (iv) *and* (v) *from Theorem* 3.1) *also hold when Eq.* (5.1) *is replaced by*

$$\mathscr{L}u + f(x, u) + f_1(x, u) = \lambda u, \tag{5.1'}$$

where $f_1: G \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions $f_1(x, \eta) = -f_1(x, -\eta)$ and $|f_1(x, \eta)| \leqslant a(x) + b|\eta|$ for every $\eta \in \mathbb{R}$ and almost all $x \in G$. Here a is a nonnegative function in $L^2(G)$ and b a positive constant.

Remark. This extension of Theorem 5.1 makes it possible to admit nonlinearities which are not monotone in η or have singularities in x. For example, consider the nonlinearity $w(x) |\eta|^\sigma \eta + a(x)^{1/2} |\eta|^\tau \sin \eta$, where w and σ satisfy (F2) and $0 < \tau \leqslant 1/2$, and $a(x)$ is an arbitrary nonnegative function in $L^2(G)$.

We prove Theorem 5.1 by verifying the hypotheses of Theorem 3.1.

LEMMA 5.1. *$(X, \| \|_X)$ satisfies hypothesis* (I).

Proof. By construction X is a uniformly convex Banach space (see Lemma 3.0). Since the C^∞-functions with compact support are contained in X we know that X is dense in $H = L^2(G)$ and that $\dim X = \infty$. It remains to show the imbedding from $X \to H$ is uniformly approximated by compact operators. Let $G_n \subseteq G$ be a sequence of bounded domains such that $\bigcup_{n=1}^\infty G_n = G$ and $\bar{G}_{n-1} \subseteq G_n$ for $n \geqslant 2$. Define $K_n: X \to H$ by $K_n u(x) = h_n(x)\, u(x)$ for every $x \in G$, where $h_n: G \to [0, 1]$ is a smooth function which is 1 on $\bar{G}_{n-1}$ and vanishes outside G_n. The operator K_n permits the factorisation

$$X \to W_0^{1,2}(G) \xrightarrow{R} W_0^{1,2}(G_n) \to L^2(G_n) \xrightarrow{E} L^2(G),$$

where R is given by the same formula as K_n and where E denotes extension by the constant 0 outside G_n. The operator K_n is compact since all factors are continuous and the imbedding $W_0^{1,2}(G_n) \to L^2(G_n)$ is compact (see Adams [1]).

Therefore it suffices to show that K_n approaches the embedding of X in H

uniformly on bounded subsets of X, as $n \to \infty$. Put $\tilde{G}_n = G - G_{n-1}$ for $n \geqslant 2$. For $u \in X$ we have

$$\|u \dot{-} K_n u\|^2 \leqslant \int_{\tilde{G}_n} u^2(x)\, dx$$

$$= \int_{\tilde{G}_n} (|u|^2 w^{-2/p} w^{2/p})(x)\, dx$$

$$\leqslant \left(\int_{\tilde{G}_n} (|u|^p w)(x)\, dx\right)^{2/p} \left(\int_{G_n} w^{-2/\sigma}(x)\, dx\right)^{\sigma/p}$$

$$\leqslant \|u\|_Y^2 \left(\int_{\tilde{G}_n} w^{-2/\sigma}(x)\, dx\right)^{\sigma/p}.$$

Hence, for every $u \in X$ we have $\|u - K_n u\| \leqslant C(n) \|u\|_X$ where $C(n) \to 0$ as $n \to \infty$ by (F2).

The nonlinear operator $F: X \to X^*$ is defined by

$$\langle F(u), h \rangle = \int_G f(x, u(x))\, h(x)\, dx \qquad (u, h \in X).$$

LEMMA 5.2. *$F: X \to X^*$ is well defined and uniformly continuous on bounded sets.*

Proof. We use the following identity, valid for all $x \in G$, ξ, $\eta \in \mathbb{R}$:

$$f(x, \xi) - f(x, \eta) = \int_0^1 g(x, \eta + t(\xi - \eta))(\xi - \eta)\, dt. \tag{5.6}$$

The identity (5.6) and (F1), (F3) yield for $u, v, h \in X$ and $y_t = u + t(v - u)$:

$$|\langle F(v) - F(u), h \rangle|$$

$$\leqslant \int_G |f(x, v(x)) - f(x, u(x))|\, |h(x)|\, dx$$

$$\leqslant \int_G \int_0^1 g(x, y_t(x))\, |u(x) - v(x)|\, |h(x)|\, dt\, dx$$

$$\leqslant \int_G \int_0^1 \{w(x)\, |y_t(x)|^\sigma + g_1(x, y_t(x))\}\, |u - v|\, (x)\, |h(x)|\, dt\, dx$$

$$\leqslant \int_G \int_0^1 w(x)\, |y_t(x)|^\sigma\, |u - v|\, (x)\, |h(x)|\, dt\, dx$$

$$+ \|u - v\|_X \|h\|_X \int_0^1 \Omega(\|y_t\|_X)\, dt.$$

Here as in the sequel changing the order of integration is justified since all integrals are bounded.

By Hölder's inequality for three factors we obtain

$$\int_G \int_0^1 w(x)\,|y_t(x)|^\sigma\,|u-v|\,(x)\,|h(x)|\,dt\,dx$$
$$= \int_0^1 \int_G \{(w^{1/p}\,|u-v|)(w^{1/p}\,|h|)(w^{\sigma/p}\,|y_t|^\sigma)\}\,(x)\,dx\,dt$$
$$\leqslant \int_0^1 \|u-v\|_Y\,\|h\|_Y\,\|y_t\|_Y^\sigma\,dt \leqslant \|u-v\|_X\,\|h\|_X \int_0^1 \|y_t\|_X^\sigma\,dt.$$

Hence, we obtain

$$|\langle F(v)-F(u), h\rangle| \leqslant \|u-v\|_X\,\|h\|_X \int_0^1 [\|y_t\|_X^\sigma + \Omega(\|y_t\|_X)]\,dt. \tag{5.7}$$

Choosing $u \equiv 0$ we see that F is well defined as an operator from X to X^*. Further we see that F is uniformly continuous on bounded sets.

LEMMA 5.3. *$F\colon X \to X^*$ is the (odd) gradient of the even Fréchet differentiable functional $\phi\colon X \to \mathbb{R}$ which is defined by*

$$\phi(u) = \int_G \int_0^{u(x)} f(x,\zeta)\,d\zeta\,dx.$$

Proof. The operator F is odd since g is even in the second variable. By Theorem 3.3 of Vainberg [25] it suffices to show that F is the Gateaux derivative of the functional ϕ. Let $u, h \in X$. For any $t \in \mathbb{R}$ and $x \in G$ we have with $\eta = u(x) + th(x)$

$$\int_0^\eta f(x,\zeta)\,d\zeta = \int_0^{u(x)} f(x,\zeta)\,d\zeta + \int_0^t f(x, u(x)+sh(x))\,h(x)\,ds,$$

and consequently

$$\phi(u+th) = \phi(u) + \int_G \int_0^t f(x, u(x)+sh(x))\,h(x)\,ds\,dx$$
$$= \phi(u) + \int_0^t \langle F(u+sh), h\rangle\,ds,$$

where the change of integration is justified since $(f(x, u(x)+sh(x))\,h(x)$ is integrable. The function $\phi(u+th)$ is differentiable in t by the last formula.

$$\frac{d}{dt}\phi(u+th)_{|t=0} = \langle F(u), h\rangle$$

i.e., F is the Gateaux derivative of ϕ.

LEMMA 5.4. *$F: X \to X^*$ is monotone and satisfies* (II(2, 3)).

Proof. Because of

$$f(x,\zeta)-f(x,\eta)=\int_\eta^\zeta g(x,\xi)\,dw(x)\int_\eta^\zeta |\xi|^\sigma\,d\xi$$

$$= w(x)(\delta/(\sigma+1))(|\zeta|^\sigma\zeta-|\eta|^\sigma\eta)$$

we obtain by using symmetry between ζ and η and the inequality $(|\zeta|^\sigma\zeta-|\eta|^\sigma\eta)(\zeta-\eta)\geqslant 2^{-\sigma}|\zeta-\eta|^{\sigma+2}$:

$$(f(x,\zeta)-f(x,\eta))(\zeta-\eta)\geqslant 2^{-\sigma}\delta(\sigma+1)^{-1}\,w(x)\,|\zeta-\eta|^{\sigma+2}. \tag{5.8}$$

Now assume $u, v\in X$ and $u_n \rightharpoonup_X u$. Application of (5.8) gives

$$\langle F(v)-F(u), v-u\rangle=\int_G (f(x,v(x))-f(x,u(x)))(v(x)-u(x))\,dx$$

$$\geqslant 2^{-\sigma}\delta(\sigma+1)^{-1}\,\|v-u\|_Y^p.$$

This relation implies that $F: X\to X^*$ is monotone, and on replacing v by u_n we also see that (II(2)) holds. Finally it is clear that (II(3)) is satisfied with $\gamma(t)=t^p$.

Lemmas 5.1, 5.2, 5.3, 5.4 show that all the hyptheses of Theorem 3.1 are satisfied, which completes the proof of Theorem 5.1.

Proof of Corollary 5.1. Under the assumed conditions f_1 generates a continuous and bounded Nemitskij operator $\tilde{F}_1: H\to H$, and this operator is the Gateaux gradient of the functional $\tilde{\phi}_1: H\to \mathbb{R}$ given by $\tilde{\phi}_1(u)=\int_G^u\int_0^{(x)} f_1(x,\eta)\,d\eta\,dx$ $(u\in H)$. But from Lemma 5.1 we know that X is compactly imbedded in H. Hence, H is continuously (even compactly) imbedded in X^*, and it immediately follows that f_1 generates a strongly continuous Nemitskij operator $F_1: X\to X^*$ and that this operator is the Gateaux derivative of the restriction ϕ_1 of $\tilde{\phi}_1$ to X. Moreover, since $\tilde{F}_1$ is bounded, the relation $\tilde{\phi}_1(u)=\int_0^1\langle \tilde{F}_1(tu), u\rangle\,dt$ $(u\in H)$ shows clearly that $\tilde{\phi}_1$ is bounded on bounded subsets of H. Finally, F_1 is obviously an odd operator, and, hence, all the conditions assumed in Theorem 3.2 are satisfied.

Finally we turn to the question of bifurcation. It turns out that under slightly more restrictive assumptions on problem (5.1), (5.2) the conditions (III) from Section 4 are satisfied. There exist several possibilities for such additional restrictions and we shall limit ourselves to one of them here. Specifically let us assume:

(F4) There exists a constant $C_1 > 0$ such that for all $x \in G$, $\eta \in \mathbb{R}$

$$\eta f(x, \eta) \leqslant C_1 \int_0^\eta f(x, \zeta)\, d\zeta.$$

(F5) For some integer $m > n/4$ the domain G is of class C^{2m} (in the sense of [2], Sect. 9), and the functions P_{jk} $(j, k = 1,..., n)$ have bounded continuous partial derivatives in G up to order $2m - 1$.

(F6) The function equal to w in G and to the constant 0 in $\mathbb{R}^n - G$ is locally integrable.

Condition (F4) does not involve a severe restriction if (F1) is satisfied. For example, it is satisfied if g_1 can be chosen zero in (F1), or if $f(x, \eta) = \sum_{j=0}^{k} w_j(x)\, |\eta|^{\tau_j}\, \eta$ with nonnegative coefficients w_j and positive exponents τ_j. Condition (F5) is a standard requirement for the application of global regularity theory. Condition (F6) means that w does not grow too strongly near ∂G.

THEOREM 5.2. *Suppose* (F1), (F2), (F3), (F4), (F5), (F6) *are satisfied. Then the assertions of Theorem* 4.1 *are valid for the problem* (5.1), (5.2). *In particular the lowest point λ^* of the spectrum of the self-adjoint operator L is always a bifurcation point.*

EXAMPLE 5.2. As the simplest example consider

$$-\Delta u + w(x)\, |u|^\sigma\, u = \lambda u \qquad (u \in W^{1,2}(\mathbb{R}^n)),$$

where w is continuous, nonnegative, and $\int_{\mathbb{R}^n} w^{-2/\sigma}(x)\, dx < \infty$. Then our theorems show that there are infinitely many global solution "branches" emanating at $\lambda^* = 0$ from the trivial line of solutions.

Proof of Theorem 5.2. We verify assumption (III) from Section 4. First of all, putting $u = 0$ in estimate (5.7) yields $|\langle F(v), h\rangle| \leqslant \|v\|_X \|h\|_X (\|v\|_X^\sigma + \int_0^1 \Omega(t \|v\|_X)\, dt)$ and hence (III(1)) since Ω is continuous and $\Omega(0) = 0$. To prove (III(2)), we use (F4) and obtain

$$\begin{aligned}\langle F(u), u\rangle = \int_G u(x) f(x, u(x))\, dx &\leqslant C_1 \int_G \int_0^{u(x)} f(x, \zeta)\, d\zeta\, dx \\ &= C_1 \phi(u)\end{aligned}$$

even for arbitrary $u \in X$.

Now, define an operator $\mathscr{L}_0$ by $\mathscr{L}_0 u = \mathscr{L}u - Qu$ for $u \in C_0^\infty(G)$ and let L_0 be the unique self-adjoint extension of $\mathscr{L}_0$ whose domain D_{L_0} is contained in $W_0^{1,2}(G)$. Since $Q \in L^\infty(G)$, we have $D_L = D_{L_0}$, and the graph norms induced by L, respectively L_0, are equivalent. Therefore, (III(3)) will be established once we have found a subset of $X \cap D_{L_0}$ which is dense in the Hilbert space H_1 constructed from D_{L_0} by introducing the graph norm of L_0 and the respective inner product. To this end, consider a function $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta|_{]-\infty,1]} = 1$, $\zeta|_{[2,\infty[} = 0$, and $0 \leqslant \zeta \leqslant 1$ on $[1,2]$, and put $h_R(x) = \zeta(|x|/R)$ for $R > 0$, $x \in \mathbb{R}^n$ where $|x|$ denotes the Euclidean norm of x. Then $h_R \in C_0^\infty(\mathbb{R}^n)$, and we have

LEMMA 5.5. *For $u \in H_1$ we have $h_R u \in H_1$ for any $R > 0$, and moreover h_R tends to u strongly in H_1 as $R \to \infty$.*

Proof. We know that $H_1 = \{u \in W_0^{1,2}(G) \mid \sum_{j,k=1}^n (P_{jk} u_{x_k})_{x_j} \in L^2(G)\}$, and hence the result follows from an easy calculation, noting that the P_{jk} and their first derivatives are bounded by assumption.

Now, let $B = (I + L_0)^{-1}$. Then B is a continuous bijection of $H = L^2(G)$ onto H_1, and B also maps H_1 continuously onto the domain of L_0^2, which is a dense subspace subspace of the Hilbert space H_1. Let us now inductively define subsets D_l, D_l^0 of H_1 by the rules:

$$
\begin{aligned}
D_1 &= H_1, \\
D_l^0 &= \{h_R u \mid u \in D_l, R > 0\} \qquad (l \geqslant 1), \\
D_{l+1} &= B(D_l^0) \qquad (l \geqslant 1),
\end{aligned}
$$

We shall prove the following assertions (where closures are always taken with respect to the norm of H_1) by induction on l

(i) $\bar{D}_l = H_1$,

(ii) $\overline{D_l^0} = H_1$,

(iii) $D_l^0 \subseteq W^{2l,2}(G)$ if $l \leqslant m$.

For $l = 1$, (i) is clear from the definition, (ii) follows from Lemma 5.5, and for (iii) note that by definition any $u \in D_1$ is a weak solution of the equation $\mathscr{L}_0 u = v$ for some $v \in L^2(G)$, and hence $u \in W^{2,2}(G_R)$ for any $R > 0$ (where $G_R = \{x \in G \mid |x| < R\}$) by a well-known global regularity theorem (see, e.g., [2, Theorem 9.8]). This yields $h_R u \in W^{2,2}(G)$ for any R, hence (iii). Suppose now that (i), (ii), (iii) are established for some $l \geqslant 1$. Then, by the continuity of B, we have $B(H_1) = B(\overline{D_l^0}) \subseteq \overline{B(D_l^0)} = \overline{D_{l+1}}$ which yields $\overline{D_{l+1}} = H_1$ since $B(H_1)$ is dense in H_1. Next, from Lemma 5.5 we infer that $D_{l+1} \subseteq \overline{D_{l+1}^0} \subseteq H_1$ and hence $\overline{D_{l+1}^0} = H_1$. Finally, any $u \in D_{l+1}$ is a weak solution in $W_0^{1,2}(G)$ of the equation $L_0 u + u = v$ for some $v \in D_l^0 \subseteq W^{2l,2}(G)$. Hence, in case $l + 1 \leqslant m$ we may again apply the above-mentioned regularity theorem from which it follows that $u \in W^{2l+2,2}(G_R)$ for any $R > 0$, which in turn yields (iii) for $l+1$.

Since $2m > n/2$, it follows from (iii) and the Sobolev imbedding theorem that any $u \in D_m^0$ is a bounded continuous function in G and, hence, an element of Y by (F6). On the other hand, $D_m^0 \subseteq H_1 \subseteq W_0^{1,2}(G)$ and, thus, D_m^0 is a subset of $X \cap H_1$ which is dense in H_1, as desired. This ends the proof of Theorem 5.2.

Remark. Clearly, requirements (F5), (F6) have only been used to establish (III(3)), and this can be done in various other ways. For example, by an obvious modification of the above proof one sees that (III(3)) (and hence Theorem 5.2) can also be proved when $m \geqslant n\sigma/(4p)$ if (F6) is slightly strengthened by assuming that any function $u \in L^p(G)$ which has bounded support belongs to Y. For the case $G = \mathbb{R}^n$ we only need $m = 1$ because functions in $D_1^0 \subseteq W_0^{2,2}(\mathbb{R}^n)$ can be mollified by convolution. Finally, in case $n = 2$ or $n \geqslant 3$, $\sigma \leqslant 4/(n-2)$ there is a Sobolev imbedding $W_0^{1,2}(G) \to L^p(G)$, and hence we obtain (III(3)) directly from Lemma 5.5 if the above-mentioned strengthening of (F6) is used. In this case, no assumptions whatever on the geometry of the domain G are needed.

Remark. The assumption $Q \in L^\infty(G)$ could be relaxed so as to include operators $L = -\Delta + Q$ of the type occurring as Hamiltonian operators in quantum mechanics. To see this, write $L = L_0 + Q$ as in the proof of Theorem 5.2 and assume that the operator generated by Q is relatively L_0-bounded with relative bound <1. Then it follows from well-known perturbation theorems that $D_L = D_{L_0}$ with equivalence of graph norms and that the quadratic form associated with L generates an equivalent norm on $W_0^{1,2}(G)$. The proofs of Theorems 5.1 and 5.2 then go through.

6. Application to Ordinary Differential Equations

It is evident that the results of the preceding section hold for the case $n = 1$, in particular. However, in this case our assumptions can be considerably relaxed due to the particularly pleasant imbedding properties enjoyed by the space X (cf. Proposition 6.1), and the present section is devoted to proving existence and bifurcation theorems similar to those of Section 5 under these relaxed assumptions, along with some additional results. We limit our treatment to the case $G =]0, \infty[$; the case $G =]-\infty, \infty[$ could be treated in a similar way.

Specifically, we shall consider the nonlinear Sturm–Liouville problem

$$-(P(x)\,u')' + Q(x)\,u + f(x, u) = \lambda u \qquad (x > 0), \tag{6.1}$$

$$u(0) = 0, \qquad u \in L^2(0, \infty), \tag{6.2}$$

where P, Q are bounded continuous real-valued functions on $[0, \infty[$ such that P is of class C^1 and $P(x) \geqslant P_0 > 0$ for every $x \geqslant 0$, and where $f: (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. Further, as a function of its second variable only, f is assumed to be odd, monotonely nondecreasing, and continuously differentiable, its derivative $\partial f(x, \eta)/\partial \eta$ being denoted by $g(x, \eta)$, as before.

We now indicate how assumptions (F1)–(F6) from Section 5 can be relaxed in the case $n = 1$. The gist of these relaxations is that the restrictions imposed by (F1), (F3), (F4) on the behaviour of $f(x, \eta)$ as function of η are now required for *small* $|\eta|$ *only*. More precisely, we assume:

($\mathrm{F1^b}$) There are continuous functions $w: (0, \infty) \to [0, \infty)$, g_1: $(0, \infty) \times \mathbb{R} \to [0, \infty)$, and ω_0, Ω_0: $[0, \infty) \to [0, \infty)$ such that

$$w(x)\, \omega_0(|\eta|) \leqslant g(x, \eta) \leqslant w(x)\, \Omega_0(|\eta|) + g_1(x, \eta) \tag{6.3}$$

for any $x > 0$, $\eta \in \mathbb{R}$, and such that for a certain $\sigma > 0$ we have

$$\liminf_{t \to 0+} \omega_0(t)\, t^{-\sigma} > 0 \tag{6.4}$$

and

$$\limsup_{t \to 0+} \Omega_0(t)\, t^{-\sigma} < \infty. \tag{6.5}$$

Finally, it is assumed that Ω_0 is monotonely nondecreasing, that $\omega_0(t) > 0$ for $t > 0$, and that $\int_0^c w(x)\, dx < \infty$ for every $c > 0$.

It is evident that the operator T and the spaces Y and X can be constructed as in Section 5, and that $H_T = W_0^{1,2}(0, \infty)$, the norms being equivalent. As in Section 5, we impose restrictions (F2), (F3) on the functions w respectively g_1, and for the bifurcation result we also need:

($\mathrm{F4^b}$) There exist constants $\delta > 0$, $C_1 > 0$ such that for every $x > 0$ and every η with $|\eta| < \delta$ we have

$$\eta f(x, \eta) \leqslant C_1 \int_0^\eta f(x, \zeta)\, d\zeta. \tag{6.6}$$

Before commenting on these new assumptions, we prove a crucial lemma.

PROPOSITION 6.1. *Every* $u \in X$ *is continuous, vanishes at* $x = 0$, *and satisfies the estimate* $|u(x)| \leqslant \varepsilon(x)\, \|u\|_X$ *for every* $x \geqslant 0$, *where*

$$\varepsilon(x) = \min\left(1, \left(\int_x^\infty w(y)^{-2/\sigma}\, dy\right)^{\sigma/4p}\right).$$

Proof. Let $u \in X$. Since $X \subseteq W_0^{1,2}(0, \infty)$, we know that u is absolutely continuous and that u vanishes at $x = 0$ and at infinity. Hence, we have

$-u(x)^2 = \int_x^\infty 2uu'\,dy$ almost everywhere in $[0, \infty)$, and thus the Hölder inequality for three factors yields $|u(x)|^2 \leqslant 2\int_x^\infty |uu'|\,dy = 2\int_x^\infty |u'| \cdot |uw^{1/p}| \cdot w^{-1/p}\,dy \leqslant 2\,\|u'\|\,\|u\|_Y\,(\int_x^\infty w^{-2/\sigma}\,dy)^{\sigma/2p}$. Since u is continuous, this estimate holds even for all $x \geqslant 0$. Together with the inequalities $2\,\|u'\| \cdot \|u\|_Y \leqslant \|u\|_X^2$ and $\sup_{x\geqslant 0} |u(x)| \leqslant \|u\|_{W_0^{1,2}(0,\infty)}$ the result follows.

Remarks. (i) It should be noted that condition (F3) amounts to a much weaker restriction in the case $n = 1$ than in the general case. To see this clearly, consider the case where g_1 can be estimated in the form

$$g_1(x, \eta) \leqslant W_1(x)\Omega_1(|\eta|) \qquad (x > 0, \eta \in \mathbb{R})$$

with continuous functions $W_1\colon (0, \infty) \to [0, \infty)$, $\Omega_1\colon [0, \infty) \to [0, \infty)$ such that $\Omega_1(0) = 0$, Ω_1 is nondecreasing, and W_1 satisfies

$$\int_0^\infty W_1(x)\,|u(x)\,v(x)|\,dx \leqslant C\,\|u\|_X\,\|v\|_X \tag{6.7}$$

for $u, v \in X$, where $C > 0$ is a constant. Then (F3) is satisfied with $\Omega = C\Omega_1$, as is immediately seen from Proposition 6.1. On the other hand, (6.7) can be ensured using the method indicated in Example 5.1. More specifically, if we choose $p_1,..., p_4 \in (1, \infty]$ such that $p_1^{-1} + p_2^{-1} + p_3^{-1} + p_4^{-1} = 1$ and p_3, $p_4 \geqslant p$ and we require

$$\int_0^\infty W_1(x)^{p_1}\,w(x)^{\alpha p_1}\,dx < \infty, \tag{6.8}$$

where $\alpha = 2(\sigma p_2)^{-1} - p_3^{-1} - p_4^{-1}$, then (6.7) holds. Namely, denoting the sup-norm by $\|\ \|_\infty$, we can write $\int_0^\infty W_1(x)\,|u(x)\,v(x)|\,dx \leqslant \|u\|_\infty^{1-p/p_3}\,\|v\|_\infty^{1-p/p_4} \cdot \int_0^\infty (W_1 w^\alpha w^{-2/\sigma p_2})\;|u|^{p/p_3}\;w^{1/p_3}\;|v|^{p/p_4}\;w^{1/p_4})(x)\,dx$ for arbitrary $u, v \in X$, and from this we easily infer (6.7), using the Hölder inequality for four factors together with (6.8), (F2), Proposition 6.1, and the imbedding $X \to Y$. The case where some of the p_k ($k = 1,..., 4$) are infinite can be dealt with by obvious modifications.

(ii) To shed some light on condition (F4$^\text{b}$), let us note that this condition is automatically satisfied if (F1$^\text{b}$) holds with $g_1 = 0$. Another interesting case is that of a nonlinearity of the form $f(x, \eta) = |\eta|^\mu f_0(x, \eta)$, where $0 \leqslant \mu < 1$, and where there exists an integer $m \geqslant 1$ such that for every $x > 0$, the function $f_0(x, \cdot)$ is of class C^{m+1} and $\partial^k f_0(x, 0)/\partial\eta^k = 0$ for $k = 0,..., m-1$, whereas $\partial^m f_0(x, 0)/\partial\eta^m \neq 0$. Since $f(x, \cdot)$ is assumed to be odd, it follows that m is odd and $\partial^m f_0(x, 0)/\partial\eta^m > 0$. Thus, we can infer (6.6) from the Taylor formula for f_0 if there are constants $\delta > 0$ and $K_1 > 0$ (independent of x) such that

$$\left| \frac{\partial^{m+1} f_0}{\partial\eta^{m+1}}(x, \eta) \right| \leqslant K_1 \frac{\partial^m f_0}{\partial\eta^m}(x, 0)$$

for $|\eta| < \delta$. Moreover, it is clear that a finite sum of terms satisfying ($F4^b$) also satisfies ($F4^b$).

THEOREM 6.1. (a) *Suppose* ($F1^b$), (F2), (F3) *are satisfied for problem* (6.1), (6.2). *Then for every* $r > 0$, *problem* (6.1), (6.2) *has an infinite sequence* (u_j^r, λ_j^r) $(j \geqslant 1)$ *of mutually distinct classical solutions such that* $\|u_j^r\|^2 = 2r$ *and* $u_j^r \in W_0^{1,2}(0, \infty)$ *for every* $j \geqslant 1$. *Moreover, assertions* (i)–(v) *of Theorem* 3.1 *are valid for these solutions.*

(b) *Suppose in addition that* ($F4^b$) *is satisfied. Then the assertions of Theorem* 4.1 *also hold for the solutions* (u_j^r, λ_j^r), *and for any* $j \geqslant 1$, *we have* $\lim_{r \to 0+} u_j^r(x) = 0$ *uniformly for* $x \geqslant 0$.

Remarks. (i) This theorem clearly generalizes results from [15] which were obtained there via an entirely different method. Moreover, it is easy to see that the solution u_1^r corresponding to the lowest critical value can always be chosen a *positive* function, and that under the additional assumption $f(x, \eta) < \eta g(x, \eta)$ for every $x \geqslant 0$, $\eta > 0$, the uniqueness and pointwise monotonicity results obtained in [15] for positive solutions can also be carried over to the present situation.

(ii) For a nontrivial solution $(u, \lambda) \in X \times [\lambda^*, \infty)$ of (6.1), (6.2) we may investigate the spectrum of the linear differential operator arising when the left-hand side of (6.1) is linearized at u, and it is an interesting question how this spectrum behaves when (u, λ) runs through a solution "branch" emanating from $(0, \lambda^*)$. However, the situation seems to be rather complicated even in the simple case of the equation

$$-u'' + w(x)\,|u|^\sigma\, u = \lambda u. \tag{6.9}$$

There are examples of functions w for which the linearization at a nontrivial solution $u \in X$ has compact resolvent and, hence, empty essential spectrum. On the other hand, in case the growth of w for $x \to \infty$ is weaker than exponential, it can be shown that there is an upper bound for the minimum of the essential spectrum of the linearization at a solution $u \in X$, $u \neq 0$ of (6.9). More precisely if $w(x) = o(e^{\alpha x})$ at infinity for every $\alpha > 0$ and if $(u, \lambda) \in X \times [0, \infty)$ is a solution of (6.9), one can show that $\liminf_{x \to \infty} w(x)\,|u(x)|^\sigma \leqslant \lambda$. Hence, it follows from well-known theorems on singular boundary value problems that $\min \sigma_e(L_u) \leqslant (\sigma + 1)\,\lambda$, where L_u denotes an arbitrary self-adjoint extension of the linearized differential operator $\mathscr{L}_u$ given by $\mathscr{L}_u h := -h'' + (\sigma + 1)\, w\,|u|^\sigma\, h$, and where $\sigma_e(L_u)$ denotes the essential spectrum of L_u.

Proof of Theorem 6.1. The proof proceeds along the same lines as in Theorems 5.1, 5.2, and we shall give details only as far as the necessary modifications are concerned. Again, our main task will consist in establishing the hypotheses of Theorems 3.1, respectively 4.1.

(a) Only (F2) and the definition of X was used to prove Lemma 5.1 and, hence, this lemma is valid under the present assumptions. The nonlinear operator F is defined as in Section 5 and we can then prove the assertions of Lemma 5.2 without difficulty. We only have to replace (5.7) by the inequality

$$|\langle F(v) - F(u), h\rangle| \leqslant \|u - v\|_X \|h\|_X \int_0^1 [C_0 \Omega_0(\|y_t\|_X) + C_1 \|y_t\|_X^\sigma + \Omega(\|y_t\|_X)]\, dt, \qquad (6.10)$$

where u, v, $h \in X$, $y_t = u + t(v-u)$, and where C_0, $C_1 > 0$ are constants which can be chosen independent of u, v, h as long as u, v range throughout some bounded subset of X.

To prove (6.10), consider $y \in X$, $M > 0$ such that $\|y\|_X \leqslant M$ and note that $|y(x)| \leqslant M \in (x)$ by Proposition 6.1. By assumption (6.5) there are constants $\delta > 0$, $C_1 > 0$ such that $0 \leqslant \Omega_0(|\eta|) \leqslant C_1 |\eta|^\sigma$ for $|\eta| < \delta$, and since $\varepsilon(x)$ is nonincreasing and tends to zero as $x \to \infty$, there exists $x_0 > 0$ such that $\varepsilon(x) \leqslant \delta/M$ for any $x \geqslant x_0$. It follows that $\Omega_0(|y(x)|) \leqslant C_1 |y(x)|^\sigma$ whenever $x \geqslant x_0$, $\|y\|_X \leqslant M$. Now let u, v, $h \in X$, $\|u\|_X$, $\|v\|_X \leqslant M$, and $0 \leqslant t \leqslant 1$. Then $\|y_t\|_X \leqslant M$, and, hence, we obtain

$$\begin{aligned}
&\int_0^\infty \Omega_0(|y_t(x)|)\, w(x)\, |u(x) - v(x)|\, |h(x)|\, dx \\
&\quad = \int_0^{x_0} + \int_{x_0}^\infty \leqslant \Omega_0(\|y_t\|_\infty) \int_0^{x_0} w(x)\, dx\, \|u - v\|_\infty \|h\|_\infty \\
&\quad + C_1 \int_{x_0}^\infty |y_t(x)|^\sigma w(x)\, |u(x) - v(x)|\, |h(x)|\, dx \\
&\quad \leqslant C_0 \Omega_0(\|y_t\|_\infty) \|u - v\|_\infty \|h\|_\infty + C_1 \|y_t\|_Y^\sigma \|u - v\|_Y \|h\|_Y \\
&\quad \leqslant [C_0 \Omega_0(\|y_t\|_X) + C_1 \|y_t\|_X^\sigma] \|u - v\|_X \|h\|_X.
\end{aligned}$$

Here we have put $C_0 = \int_0^{x_0} w(x)\, dx$, and we have used the assumptions that Ω_0 is nondecreasing and that $C_0 < \infty$ along with $\varepsilon(x) \leqslant 1$ and the Hölder inequality for three factors. Now (6.3) yields $\int_0^\infty g(x, y_t(x))\, dx \leqslant [C_0 \Omega_0(\|y_t\|_X) + C_1 \|y_t\|_X^\sigma] \cdot \|u - v\|_X \|h\|_X + \int_0^\infty g_1(x, y_t(x))\, dx$, and, thus, we can infer (6.10) from (5.6) with the aid of (F3) in the same way as was done for (5.7) in the proof of Lemma 5.2.

The proof of Lemma 5.3 also goes through when the appeal to (5.7) is replaced by an appeal to (6.10). Adapting the proof of Lemma 5.4, we first arrive at the inequality

$$(f(x,\zeta)-f(x,\eta))(\zeta-\eta)\geqslant\frac{2^{-\sigma}}{\sigma+1}\,\delta(M)\,|\zeta-\eta|^{\sigma+2}\,w(x) \tag{6.11}$$

for $|\zeta|$, $|\eta|\leqslant M$, $x>0$, where we have put $\delta(M)=\inf_{0\leqslant t\leqslant M}\omega_0(t)\,t^{-\sigma}$. Note that $\delta(M)>0$ for every M by virtue of (6.4) and the assumption that ω_0 is continuous and positive on $(0,\infty)$. We can now derive (II(2)) as in the proof of Lemma 5.4, replacing (5.8) by (6.11) and taking into account that any weakly convergent sequence in X is bounded in $L^\infty(0,\infty)$. Moreover, if $B\subseteq X$ is bounded in $H_T=W_0^{1,2}(0,\infty)$, then B is also bounded in $L^\infty(0,\infty)$, say $\|u\|_\infty\leqslant M$ for every $u\in B$.

From (6.11) it then follows that $\langle F(u),u\rangle\geqslant 2^{-\sigma}(\sigma+1)^{-1}\,\delta(M)\,\|u\|_Y^p$ for every $u\in B$, and this is clearly enough to prove Lemma 3.2. We cannot establish (II(3)), but this is not necessary since (II(3)) has only been used to prove Lemma 3.2. Thus, all the assertions of Theorem 3.1 are valid in the present situation. The solutions (u_j^r,λ_j^r) thus obtained are also classical by classical regularity theorems of the calculus of variations since the u_j^r are in $W_0^{1,2}(0,\infty)$ and the data are assumed to be continuous.

(b) It is clearly enough to establish assumptions (III). The statement on uniform convergence follows from (4.1) and Proposition 6.1. However, since Ω_0 is continuous and $\Omega_0(0)=0$, (III(1)) obviously follows from (6.10), and for the proof of (III(2)), recall that the usual norm on $W_0^{1,2}(0,\infty)=H_T$ is equivalent to the graph norm of T, so that we can write

$$\begin{aligned}\|u\|_\infty^2&\leqslant C_0^2(\|Tu\|^2+\|u\|^2)\\&=2C_0^2(\psi(u)-\phi(u))+C_0^2\,\|u\|^2\leqslant C_0^2(2\psi(u)+\|u\|^2)\end{aligned}$$

for $u\in X$, with C_0 a positive constant. Thus, for any $u\in X$ such that $\|u\|$ and $\psi(u)$ are sufficiently small we have (6.6) for $\eta=u(x)$ and $x\geqslant 0$ arbitrary, and hence we have (III(2)). As for (III(3)), note that the linear part of (6.1) is in the limit point case at infinity by virtue of the boundedness of P and Q. This implies that the set $D_0=\{u\in D_L\mid u$ has compact support in $[0,\infty)\}$ is dense in D_L with respect to the graph norm of L. On the other hand, $D_0\subseteq H_T=W_0^{1,2}(0,\infty)\subseteq\{u\in L^2(0,\infty)\mid u$ continuous, $u(0)=0\}$, and hence the integrability requirement on w stated at the end of ($\mathrm{F1}^{\mathrm{b}}$) implies $D_0\subseteq Y$, so that we finally get $D_0\subseteq X$, i.e., we have established (III(3)). This completes the proof of Theorem 6.1.

Finally, we again use Theorem 3.2 to relax the monotonicity and sign conditions on the nonlinearity. Thus, we replace Eq. (6.1) by the equation

$$-(P(x)\,u')'+Q(x)\,u+f(x,u)+f_1(x,u)=\lambda u \tag{6.12}$$

considering it together with the boundary conditions (6.2) and retaining the assumptions of P, Q, f mentioned at the beginning of this section. Clearly Corollary 5.1 and its proof are still valid when (F1) is replaced by ($F1^b$). But here again, Proposition 6.1 enables us to permit much weaker growth restrictions for f_1.

THEOREM 6.2. *Suppose f satisfies conditions* ($F1^b$), (F2), (F3), *and suppose that f_1: $[0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f_1(x, -\eta) = -f_1(x, \eta)$ for every $x \geqslant 0$, $\eta \in \mathbb{R}$. Moreover, assume that for arbitrary $(x, \eta) \in [0, \infty) \times \mathbb{R}$ the following estimates hold.*

$$|f_1(x, \eta)| \leqslant w_1(x)\, \Omega_1(|\eta|) \tag{6.13}$$

and

$$\eta f_1(x, \eta) \geqslant -a(x)\,|\eta| - b\eta^2, \tag{6.14}$$

where $a \in L^2(0, \infty)$ is nonnegative almost everywhere, $b \geqslant 0$ is a constant, Ω_1: $[0, \infty) \to [0, \infty)$ is nondecreasing and tends to zero at the origin, and where w_1: $[0, \infty) \to [0, \infty)$ is measurable and satisfies

$$\int_0^\infty w_1(x)\,|h(x)|\,dx \leqslant \text{const}\,\|h\|_X \tag{6.15}$$

for every $h \in X$.

Then assertions (i), (ii), (iii) *of Theorem* 3.1 *hold for the problem* (6.12), (6.2), *and the solutions guaranteed by that theorem are classical solutions of* (6.12).

Remark. Sufficient conditions for (6.15) in terms of explicit growth requirements on w_1 can easily be derived in the same way as has been done for (F3) before. Two particular such conditions are (i) $w_1 \in L^2(0, \infty)$ or (ii) $\int_0^\infty w_1^{1+1/(\sigma+1)} w^{-1/(\sigma+1)}(x)\,dx < \infty$.

Proof of Theorem 6.2. We shall show that f_1 generates a Nemitskij operator $F_1: X \to X^*$ having properties (i)–(iii) from Theorem 3.2. To this end, consider $u, h \in X$. Since $u \in L^\infty(0, \infty)$ and Ω_1 is nondecreasing, there is $C_1 > 0$ such that $\Omega_1(|u(x)|) \leqslant C_1$ for every $x \geqslant 0$. Thus, using (6.13) and (6.15), we obtain

$$\int_0^\infty |f_1(x, u(x))|\,|h(x)|\,dx \leqslant C_1 \int_0^\infty w_1\,|h|\,dx \leqslant \text{const}\,\|h\|_X.$$

This shows that f_1 generated a Nemitskij operator F_1 which maps X to X^*.

Next, we are going to show that F_1 is strongly continuous. Thus, consider a sequence $(u_n) \subseteq X$ which converges weakly in X to a function u. We then have to show

$$\lim_{n\to\infty} \sup_{\|h\|_X \leqslant 1} |\langle F_1(u_n) - F_1(u), h\rangle| = 0. \tag{6.16}$$

Let $\varepsilon > 0$. Since the weakly convergent sequence (u_n) is bounded in X, there is $M > 0$ such that $|u_n(x)|, |u(x)| \leqslant M\varepsilon(x)$ for every $x \geqslant 0$, $n \in \mathbb{N}$ where $\varepsilon(x)$ is as in Proposition 6.1. Since $\varepsilon(x) \to 0$ as $x \to \infty$, our assumptions on Ω_1 imply that $\lim_{x\to\infty} \Omega_1(M\varepsilon(x)) = 0$. Hence, there is $x_0 > 0$ such that

$$\Omega_1(M\varepsilon(x)) \leqslant \varepsilon/2C \qquad \text{for } x \geqslant x_0, \tag{6.17}$$

where C is the constant appearing in (6.15). Thus, for any $h \in X$ we get

$$\begin{aligned}
&|\langle F_1(u_n) - F_1(u), h\rangle| \\
&\quad \leqslant \int_0^{x_0} |f_1(x, u_n(x)) - f_1(x, u(x))|\, |h(x)|\, dx \\
&\qquad + \int_{x_0}^{\infty} \Omega_1(|u_n(x)|)\, w_1(x)\, |h(x)|\, dx + \int_{x_0}^{\infty} \Omega_1(|u(x)|)\, w_1(x)\, |h(x)|\, dx \\
&\quad \leqslant S_n x_0 \|h\|_\infty + 2\Omega_1(M\varepsilon(x_0)) \int_{x_0}^{\infty} (w_1 |h|)(x)\, dx,
\end{aligned}$$

where $S_n = \sup_{0 \leqslant x \leqslant x_0} |f_1(x, u_n(x)) - f_1(x, u(x))|$. Applying (6.15) and (6.17) we obtain

$$|\langle F_1(u_n) - F_1(u), h\rangle| \leqslant S_n x_0 \|h\|_\infty + \varepsilon \|h\|_X \leqslant (S_n x_0 + \varepsilon) \|h\|_X \tag{6.18}$$

for any $h \in X$. On the other hand, there is a compact imbedding $W_0^{1,2}(0, x_0) \to C[0, x_0]$, and the restriction defines a bounded operator $X \to W_0^{1,2}(0, x_0)$. Hence, (u_n) tends to u in $C[0, x_0]$. Since $\varepsilon(x) \leqslant 1$, we have $u(x)$, $u_n(x) \in [-M, M]$ for $0 \leqslant x \leqslant x_0$ and $n \in \mathbb{N}$. On the compact set $[0, x_0] \times [-M, M]$ the continuous function f_1 is uniformly continuous, and hence the uniform convergence of (u_n) to u on $[0, x_0]$ implies that of $(F_1(u_n))$ to $F_1(u)$. This clearly means that $S_n \to 0$ as $n \to \infty$. We, thus, obtain (6.16) from (6.18) and the fact that $\varepsilon > 0$ was arbitrary.

Clearly F_1 is odd, and the fact that it is a Gateaux gradient is proved exactly as in Lemma 5.3. It only remains to prove condition (iii) from Theorem 3.2. But, from (6.14) we obtain

$$\begin{aligned}
\langle F_1(tu), u\rangle &= \frac{1}{t} \int_0^{\infty} f_1(x, tu(x))\, tu(x)\, dx \\
&\geqslant -\int_0^{\infty} a(x)\, |u(x)|\, dx - bt \int_0^{\infty} |u(x)|^2\, dx \\
&\geqslant -\|a\|\, \|u\| - bt\, \|u\|^2 \qquad \text{for } u \in X \text{ and } 0 < t \leqslant 1,
\end{aligned}$$

and this yields

$$\phi_1(u)=\int_0^1 \langle F_1(tu), u\rangle\, dt \geqslant -\|a\|\,\|u\|-\frac{b}{2}\|u\|^2,$$

from which (iii) clearly follows.

Remark. Note that a finite sum of operators satisfying conditions (i)–(iii) from Theorem 3.2 also satisfies these conditions. Thus, we also may combine perturbations of the kind considered in Corollary 5.1 with perturbations satisfying the assumptions of Theorem 6.2. Consequently, it is possible to obtain existence results for the case where $f_1(x,\eta)$ has a singularity at some $x_0 \geqslant 0$ even when its growth for $\eta \to \infty$ is only restricted by (6.13), (6.15). We simply use a partition of unity to write f_1 in the form $f_1 = f_{11} + f_{12}$, where f_{11} is as in Corollary 5.1 and f_{12} is a Theorem 6.2.

Acknowledgment

After completion of the paper we became aware of a recent preprint entitled "Bifurcation from the essencial spectrum for odd variational operators" by V. Benci (University of Pisa) and D. Fortunato (University of Bari). These authors use the Ljusternik–Schnirelman theory in the unconstrained case to prove a bifurcation theorem which is applied to $-\Delta u + w(x)|u|^\sigma u = \lambda u$ $(u \in W_0^{1,2}(\mathbb{R}^n))$. It seems that they are the first to apply the Ljusternik–Schnirelman theory on the weighted Sobolev space X.

References

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
2. S. Agmon, "Lectures on Elliptic Boundary Value Problems," Van Nostrand–Reinhold, Princeton, N. J., 1965.
3. N. W. Bazley and T. Küpper, Branches of solutions in nonlinear eigenvalue problems, *in* "Applications of Nonlinear Analysis in the Physical Sciences" (H. Amann, N. Bazley, and K. Kirchgässner, Eds.), London 1981.
4. H. Berestycki and P.-L. Lions, Existence d'Ondes solitaires dans des problèmes nonlinéaires du type Klein–Cordon, *C. R. Acad. Sci. Paris, Ser. A* **288** (1979).
5. M. S. Berger, On the existence and structure of stationary states for a nonlinear Klein–Gordon equation, *J. Funct. Anal.* **9** (1972), 249–261.
6. M. S. Berger and M. Schechter, Imbedding theorems and quasi-elliptic boundary value problems for unbounded domains, *Trans. Amer. Math. Soc.* **172** (1972), 261–278.
7. A. Bongers, "Behandlung verallgemeinerter nichtlinearer Eigenwertprobleme mit Ljusternik–Schnirelman-Theorie," Dissertation, Universität Mainz, 1979.
8. A. Bongers, Über ein Rayleigh–Ritz-Verfahren zur Bestimmung kritischer Werte, *in* "Bifurcation Problems and their Numerical Solution," ISNM 54 (H. D. Mittelmann, H. Weber, Eds.), pp. 74–91, Birkhäuser, Basel, 1980.

9. F. Browder, Existence theorems for nonlinear partial differential equations, *in* "Global Analysis, AMS Proc. Symp. Pure Math. XVI, pp. 1–61, Amer. Math. Soc., Providence, R. J., 1970.
10. R. Chiappinelli and C. A. Stuart, Bifurcation when the linearized problem has no eigenvalues, *J. Differential Equations* **30** (1978), 296–307.
11. P. Clément, "Eigenvalue Problems for a Class of Cyclically Maximal Monotone Operators," MRC Technical Report, 1976.
12. C. Coffman, Ljusternik–Schnirelman theory and eigenvalue problems for monotone potential operators, *J. Funct. Anal.* **14** (1973), 237–252.
13. M. M. Day, Some more uniformly convex spaces, Bull. Amer. Math. Soc. **47** (1941), 504–507.
14. H.–P. Heinz, Über das Verzweigungsverhalten eines nichtlinearen Eigenwert-problems in der Nähe eines unendlich vielfachen Eigenwertes der Linearisierung, *Manuscripta Math.* **15** (1976), 105–132.
15. T. Küpper, The lowest point of the continuous spectrum as a bifurcation point, *J. Differential Equations* **34** (1979), 212–217.
16. T. Küpper, On minimal nonlinearities which permit bifurcation from the continuous spectrum, *Math. Methods Appl. Sci.* **1** (1979), 572–580.
17. T. Küpper and D. Riemer, Necessary and sufficient conditions for bifurcation from the continuous spectrum, *Nonlinear Anal.* **3** (1979), 555–561.
18. E. S. Noussair, On the existence of solutions of nonlinear elliptic boundary value problems, *J. Differential Equations* **34** (1979), 482–495.
19. P. Rabinowitz, Variational methods for nonlinear eigenvalue problems, *in* "Eigenvalues of Nonlinear Problems," C.I.M.E. (G. Prodi, Ed.), pp. 141–195, Edizioni Cremonese, Roma, 1974.
20. P. Sarreither, "Zur Verzweigung an isolierten Eigenwerten unendlicher Vielfachheit," Preprint 34, Mathematische Institute der Universität Würzburg, 1978.
21. C. A. Stuart, Bifurcation pour des problèmes de Dirichlet et de Neumann sans valeurs propres, *C. R. Acad. Sci. Paris* **288** A (1979), 761–764.
22. C. A. Stuart, Bifurcation for variational problems when the linearization has no eigenvalues, *J. Functional Anal.* **38** (1980), 169–187.
23. C. A. Stuart, Bifurcation for Dirichlet problems without eigenvalues, *Proc. London Math. Soc.* **45** (1982), 169–192.
24. C. A. Stuart, Bifurcation for Neumann Problems without eigenvalues, *J. Diff. Eq.* **36** (1980), 391–407.
25. M. M. Vainberg, "Variational Methods for the Study of Nonlinear Operators," Holden-Day, San Francisco, 1964.